Proceedings, 11th FIG Symposium on Deformation Measurements, Santorini, Greece, 2003.

A Study on Using Bayesian Statistics in Geodetic Deformation Analysis

Mualla Yalçınkaya and Emine Tanır

Karadeniz Technical University (KTU), Department of Geodesy and Photogrammetry, 61080, *Trabzon, Turkey*

Abstract

During the last decades technological progress has also affected geosciences and the observational methods in all fields of geosciences have changed completely. Therefore, surveys made for deformation detection have been conducted commonly by satellite based techniques. Consequently, deformation detection studies have been conducted in adequate accuracy in less time for larger areas. However, the increasing observational accuracy requires adequate mathematical and statistical models. Measurement errors in surveys occur no matter how measurements are taken by terrestrial and satellite techniques. These measurement errors can be interpreted as deformations if they can't be eliminated from measurements. Determining measurement errors by effective measurement analysis is as important in deformation detection studies as determining the deformation model. There are two approaches to statistics: frequentist (classical) and Bayesian. They use different definitions of probability (entailing philosophical disagreements), but in many simple cases give answers that are similar. Frequentist statistics has been used as common, partly for computational reasons. In the last decade, attention has shifted towards Bayesian statistics, which has advantages in complex problems and better reflects the way scientists think about evidence. Recently, The Bayesian Statistics has been used efficiently in the areas of engineering, social sciences and medicine.

In this study, it is aimed to investigate using Bayesian Statistics in data uncertainty analysis in geodetic deformation analysis problem. For this aim, it is introduced some theoretical background information about Bayesian theory and Bayesian statistics. And then, Bayesian and classical statistics are compared each other. Finally, it is mentioned some Bayesian applications in Geodesy.

1. Introduction

In conventional deformation analysis model, object point positions are fixed by geodetic observations at different epoch of time. When you inspect data, prior to more formal analysis, you may find that one or more values are far from the majority of observations. These unusual observations are called outliers. In such a work, it is important to get an infinity precise and accurate measurement, immune from errors. The purpose of these measurements is to determine the true value of point positions at each epoch. After some pre-processing of observations, point positions are determined. If deformation occurs, there exist point position differences between epochs. In this model, analysis is usually based on least square estimation and on statistical test. Thereby a correctly designed deformation model and normally distributed observations are assumed. In geodetic deformation analysis, point differences are because of not only object deformations but also uncertainties in data. If observational (measurement) errors exist, and can't be eliminated from measurements, these errors can be interpreted as deformations. Therefore, determining measurement error by effective analysis techniques is as important in deformation detection studies as determining the deformation model. Uncertainties arise from variations in the result of repeated observations under identical conditions and can be avoided by large number of observations**.** However, sometimes it is difficult to collect large number of observations by means of time and economic conditions. In such a situation, an efficient uncertainty assessment technique should be used for decreasing of uncertainty effect from observations. So, it can be said that uncertainty assessment is the most important study in deformation analysis studies (Caspary et al, 1986; Caspary et al, 1990, Hekimoglu et al, 2002; D' Agostini, 1995). There are two general approaches used for handling of outliers (uncertainties in data). The first is to identify the outliers and remove or edit them from data. The second is to accommodate these outliers by using some "robust" procedure which diminishes their influence in the statistical analysis (Robiah et al, 2001).

There is an intense expansion in the use of Bayesian methods in all fields of human activity that generally deal with uncertainty, including engineering, computer science, economics, medicine and even forensics (Kadone and Schum, 1996). In practice of science we constantly find ourselves in a state of uncertainty. Let us consider the outcome of you measuring the distance to mention about uncertainty in data. There are some values of instrument display you are more confident to read, others you expect less, and extremes you do not believe all. Give two events E_1 and E_2. You might consider E_2 much more probable than E_1, just meaning that you believe E_2 to happen more than E_1. You would write $P(E_2)>P(E_1)$. On the other hand, we would rather state the opposite, i.e. $P(E_1)>P(E_2)$. The reason is simply, because we don't share the same status of information. You and we are uncertain about the same event, but in a different way. Values that might appear very probable to you now, appear improbable, though not impossible, to us. In this example we introduced two crucial aspects of the Bayesian approach: The term probability has the intuitive meaning of "the degree of belief that an event will occur". Probability depends on our state of knowledge, which is usually different for different people. In other words, probability is unavoidable subjective (D'Agostini, 2003).

In this study, it is introduced some background information about Bayesian statistics and compared Bayesian and classical statistics in some aspects and finally introduced some Bayesian statistics applications for the aim of investigating of using Bayesian statistics in data uncertainty assessment in a geodetic deformation analysis problem.

2. Uncertainty in Geodetic Data

Some uncertainties arise from measurement errors. It is obvious there is no observation that is absolute free of errors. This means that if we measure some quantity and then, repeat the measurement, we will almost certainly measure a different value the second time. The only certainty is that all measurements contain some uncertainty because we make errors and tools have limits. So, we can't know true value of quantity. Even though many parameters could be measurable up to any desired precision, at least in principle, there are often significant uncertainties associated with their estimates. Uncertainty is measured with accuracy how close to the true value and precision how close to each other. (Carlson, 2002).

Uncertainties of the data are due to random selection of data, the random variability of the data, imprecision of the observation procedure and instruments, lacking reliability of the data, reduced credibility of data, data gaps and lacking consistency of data coming from different sources (Kutterer, 2001a; Kutterer, 2001b; Shylion, 2001). The uncertainty in data is based on three classes of errors: Gross errors have to be avoided or detected by control methods. Systematic errors often referred to as errors of known sources such as operator, instrument and weather conditions. However, several techniques (suitable observation configuration) are supposedly being used to eliminate or minimize them. In the classical assessment tools for geodetic data analysis, and after minimizing the systematic errors, the remaining effects are assumed to be small. They are considered (assumed) as random. Random errors occurrence is assumed for each observation. They can be treated by statistical methods. The most efficient errors on result information of geodetic network are gross errors which show faults in measures. If these errors are not eliminated from the first measurements, there exist estimation errors in many estimation procedures in this geodetic model. In order to get rid of these types of errors in geodetic problems, these errors have to be eliminated from geodetic measurement model and remeasured. One technique to identify gross errors in geodetic problem is to use appropriate techniques which determine measurement with gross error during the measurement time. However, gross errors can not be eliminated perfectly. There can exist outlier measurement close to random error and considered gross error in terms of value (Dilaver, 1996).

3. Mathematical Theories for Uncertainty Analysis

Mathematical theories for uncertainty assessment can be separated into three types. These theories are more or less based on the theory of probability and into theories which are not according to (Kuttterer 2001b).

The approximation theory is the basic mathematical theory to combine data and model. Only small and unclassified uncertainties are considered to explain the differences between data and model. The objective function is selected as a suitable function between model and data to minimize approximation errors. Least Squares Estimation (LSE) methods used in Geodesy and other sciences is the most fundamental methods for uncertainty analysis problems. This theory allows reducing the effect of outliers on the estimation of parameters. However, there are also other methods used in uncertainty analysis. Selecting different objective functions leads to robust estimation and robust statistics. Maximum Likelihood Estimation (MLE) is one of the robust estimation methods.

In probabilistic theories, uncertainty is modelled by means of random variables. The Bayes theory is probabilistics based theory and allowing the use of stochastics (sometimes subjective prior knowledge). Evidence theory or Dempster theory or theory of Hints are synonymous of the Bayesian Theory.

Non-probabilistics theories are interval mathematics, fuzzy theory, possibility theory, the theory of rough sets of artificial neural networks (Kutterer, 2001b; Chen et al, 1999; Soukup, 2001).

3.1. Probabilistic Theory for Uncertainty Analysis

Scientists often make several independent measurements of a single quantity. They obtain a distribution of measurements. These measurements may be used with probability theory to estimate the magnitude of the uncertainty of measurements. Since the standard deviation represents the range over which measurements vary, it is customary to say that the standard deviation equals the magnitude of the uncertainty of the measurements. In order to obtain reliable inference results in a geodetic problem, we have to make a precisely uncertainty assessment and determine magnitude of errors in measurements. For this aim, we should use the most suitable method in uncertainty assessment.

In probabilistic approach, the uncertainties associated with model inputs are described by probability distributions, and the objective is to estimate the outputs probability distributions. The idea in this paradigm is to say that any uncertainty can be modeled in a probabilistic way. Determination of probability distributions is accomplished by using either statistical estimation techniques or expert judgments. Statistical estimation techniques involve estimating probability distributions from available data or by collection of a large number. In cases where limited data are available, an expert judgment provides the information about the input probability distribution. There are some probability distributions commonly used in the uncertainty analysis in different fields (i.e. normal, gamma, lognormal, exponential, weibul). Normal distribution is typically used to describe unbiased measurement errors (Isukapolli, 1999).

There are two interpretations about probability. A probability is a property of a set of events in terms of frequency interpretation (objectivist, frequentist or classical). On the other hand, a probability is an expression of a person’s degree of belief regarding the truth of a proposition or the occurrence of an event in terms of subjective interpretation (subjectivist approach).

According to objectivists, probabilities are real aspects of the world that can be measured by relative frequencies of outcomes of experiments. To use frequency as a measurement of probability we have to assume that the phenomenon occurred in the past, and will occur in the future, with the same probability. But nobody can say that this hypothesis is correct. We have to guess in every single case. We have to conclude that if we want to make use of these statements to assign a numerical value to probability, we need a better definition of probability. And according to subjectivists; probabilities are descriptions of an observer's degree of belief or uncertainty rather than having any external significance. When it is impossible to state firmly if an event is true or false, we just say that this is possible, probable. Different events may have different levels of probability, depending whether we think that they are more likely to be true or false. The concept of probability is then simply a measure of the degree of belief that an event will occur. Bayesian approach is based on subjective definition of probability (Dean, 2002; Crovelli, 2000; Bod, 2001).

4. Bayes Theory

Bayes theory is used in statistical inference to update estimates of the probability that different alternatives will be true, based on observations and knowledge of how likely those observations are given each alternative. Bayesian approach is the natural one for data analysis in the most general sense, and for assigning uncertainties to the results of measurements. This information describe a general model for treating uncertainties originating from random and systematic errors in a consistent way (Bernardo and Smith, 1994; Pres, 1989).

Bayesian statistics works with conditional probability only, since a statement depends in general on the question whether a further statement is true. Therefore, it is always conditional on given circumstances, empirical or theorethical information. Let A and B be events in a random experiment with $P(B) > 0$. If A and B are two related (dependent) events the fact that A has occurred will alter the probability that B occurs. One writes A/B to denote the situation that the statement A is true under the condition that B is true and P(A/B) (conditional probability) to denote probability of A/B. This probability is well suited to express empirical knowledge, since the statement B expresses existing knowledge and A/B further knowledge (Koch, 2000a). An appropriate definition of conditional probability when all outcomes are equally likely, is given by

$$P(A|B) = \text{number of elements of } A \cap B \, / \, \text{number of elements of } B \quad (1)$$

According to equation (1), the conditional probability of A given B is defined to be

$$P(A / B) = \frac{P(A \cap B)}{P(B)} \quad (2)$$

and the conditional probability of B given A is defined to be

$$P(B / A) = \frac{P(A \cap B)}{P(A)} \quad (3)$$

P(A) is the probability that A will occur; P(B) (not used so far) is the probability that B occurs; $P(A \cap B)$ is the joint probability that both A and B occur. According to equations (2) and (3), we can write

$$P(A/B)\, P(B) = P(B/A).\, P(A) \quad (4)$$

We can rearrange equation (4) to give Bayes theory formulation,

$$P(A/B) = [\, P(B/A).\, P(A)] \, / \, P(B) \quad (5)$$

Bayesian statistics provide a conceptually simple process for updating uncertainty in the light of evidence. Initial beliefs about some unknown quantity are represented by a prior distribution. Information in the data is expressed by the likelihood function. The prior distribution and the likelihood function are then combined to obtain the posterior distribution for the quantity of interest. The posterior distribution expresses our revised uncertainty in light of the data. Within the Bayesian framework the estimation of parameters and their associated uncertainty can be addressed (Gelman and Carlin, 1995). Bayesian inference is determined as a process of fitting a probability model to a set of data and summarizing the result by a probability distribution on the parameters of the model and on unobserved quantities such as predictions for new observations. By Bayesian data analysis, we mean practical methods for making inferences from data using probability models for quantities we observe and for quantities about which we wish to learn.

Suppose that $x = (x_1, x_2, ..., x_n)$ is a vector of n observations whose probability distribution P(x/θ) depends on the values of k parameters; $\theta' = (\theta_1, \theta_2, ..., \theta_k)$ Suppose also that θ itself has a probability distribution P(θ). Bayes' theory tells us that the probability distribution for θ posterior to the data x is propositional to the product of the distribution for θ prior to the data and likelihood for θ given x. That is

$$\text{Posterior distribution } \alpha \text{ likelihood x prior distribution} \tag{6}$$

$$P(\theta|x) = P(x|\theta)\, P(\theta) \,/\, P(x) \tag{7}$$

Bayes theory for a discrete random variable and continuous variable respectively;

$$P(\theta / x) = \frac{P(x/\theta)P(\theta)}{\sum_i P(x/\theta_i)P(\theta_i)} \tag{8}$$

$$f(\theta / x) = \frac{f(x/\theta)f(\theta)}{\int f(x/\theta)f(\theta)d\theta} = \frac{f(x/\theta)f(\theta)}{\text{Total area under } f(x/\theta)} \tag{9}$$

In equation (7) and (8), P(θ) tells us what is known about θ without knowledge of data, is called the prior distribution of θ, or distribution of θ a priori. Given the data x, P(x/θ) in called as likelihood function through which the data x modify prior knowledge of θ. It can therefore be regarded as representing the information about θ coming from the data. Correspondingly, P(θ/x), which tells us what is known about θ given knowledge of data, is called the posterior distribution of θ given x, or the distribution of θ a posteriori. In equation (9) we define f(θ) as the prior probability density of the parameter θ, f(x/θ) as the conditional density of the data x given the value of θ and f(θ/x) posterior probability density of the parameter θ given the value of data x. (Gundlich and Koch, 2002; Koch, 2000; D'Agostini, 1995; Box and Tiao, 1992; Berger, 1985).

Fig. 1 shows relationship between posterior and prior distributions and likelihood functions. Probability density functions in (9) are modeled with different distributions, i.e., Gaussian, Binomial, Poission.

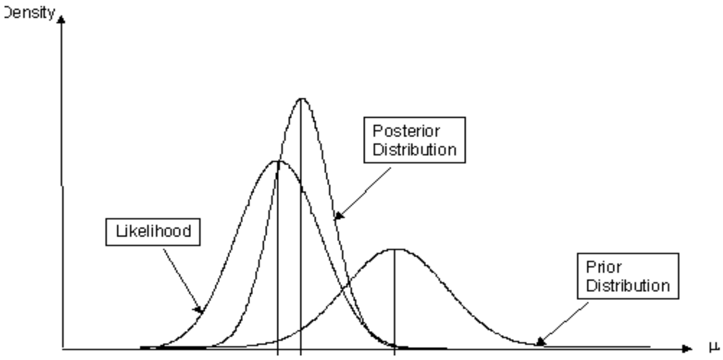

Fig. 1 Relationship between posterior and prior distributions and likelihood functions (Zeitler, 1999).

4.1. Comparision of Bayesian Statistics with Classical Statistics

Bayesian method introduces an explicit framework for incorporating prior knowledge into an analysis. In Bayesian statistics not only likelihood function (formed by data) but also prior distributions of data are used. The aim of Bayesian statisticians is to define probability distribution of θ by using knowledge and experience about θ, before getting the observation from f(x/θ) function. Information is put on analysis by P(θ) probability density function even the information is invaluable. On the other hand in classical statisticians don't accept these knowledge and experiences because of not observed and subjective. In a statistical inference problem classical statisticians infer θ unkown parameter rely on observation coming from f(x/θ) function. That is, only likelihood functions are used (see Fig. 2). In Bayesian approach, likelihood distributions are the same at each situation but prior distributions are different from each other according to prior knowledge and expert people opinion using this information because different people get different results from the same problems. This expression shows the subjective definition of probability as "degree of belief" on Bayesian statistics. Prior distributions and posterior distributions relying on prior distributions are formed completely according to expert opinions. This property is the most important property in differences between Bayesian and classical statistics.

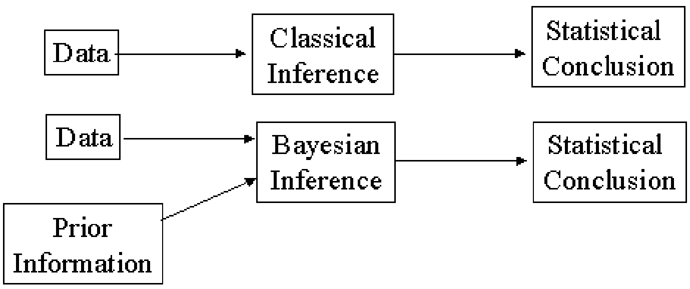

Fig. 2. Difference between Classical and Bayesian Statistics (Zeitler, 1999).

In classical statistics μ (the unknown parameters) is fixed and data are random. However, in Bayesian statistics, data (once observed) are fixed and parameter μis random and has a distribution as (10). This property is formulated in Gaussian case as equation (10). The aim in here is to estimate μ by the posterior mean and uncertainty by the posterior standart deviation.

$$P(\theta / \mu = \mu_0, \sigma^2 = \sigma_0) \approx N(\mu_0, \sigma_0^2) \tag{10}$$

Here, μ and σ has distribution respectively $P(\mu) \approx N(\tau, \gamma^2)$ and $P(\sigma) \approx N(\pi, \varepsilon^2)$. If only μ is unknown and μ_0 and σ_0 is known, according to (9) equation, main steps of Bayesian estimation as following.

$$P(\mu / D) = P(\mu / x_1, x_2, ..., x_k) = \frac{P(D/\mu)P(\mu)}{\int P(D/\mu)P(\mu)d\mu}$$

$$= \alpha \prod_{k=1}^{n} P(x_k / \mu)P(\mu) \qquad (11)$$

Here x_1, x_2, …, x_n data in sample D, $P(\mu) \approx N(\mu_0, \sigma_0^2)$ prior distribution for μ , and $P(x/\mu) \approx N(\mu, \sigma^2)$ likelihood function. Probability distribution of μ is

$$P(\mu / D) \approx N(\mu_n, \sigma_n^2) \qquad (12)$$

In the LSE, only estimated values and their covariance matrixes are obtained. In such an estimation process, statistical distributions of result information don't exist (Koch, 1999). Applying Bayesian Statistics results are found which beyond the ones of classical statistics. For instance, for robust parameter estimation not only the robust estimation can be derived but also the confidence regions for the parameters are obtained or hypotheses for the parameters cab be tested. The latter results are not available in classical statistics. Similarly the confidence regions for variance components and the tests of variance components have been derived by Bayesian statistics (Koch, 2000; Gundlich and Koch, 2002**)**

In classical methods, the random variable associated with the stochastic part has a normal distribution. The principle of minimizing the total of residual squares is equal to the principle of maximization of normal distribution. In Bayesian statistics, arbitrary probability distributions of measurements and design equations are theoretically acceptable to produce correct probability distributions of estimated parameters and adjusted measurements. If probability distribution of measurements is normal and functions in design equations are linear, the resulting probability distributions of estimated parameters and adjusted measurements are the same as those obtained by the classical least squares method.

In Bayesian approach, having the probability distribution is first (determination of probability distribution of estimated parameters and/or adjusted measurements), point estimation, interval estimation, hypothesis testing, assessment of accuracy and precision can be obtained in the second step (Soukup, 2001**)**. Difficulty of Bayesian approach is fitting probability distribution to data and making inferences from parameter distributions. These processes make more complex this approach than classical methods.

5. Some Applications of Bayesian Statistics in Geodesy

Bayesian approach in measurement uncertainty is universally applicable for the most measurement data evaluation tasks including complex nonlinear adjustments and, in particular, in cases where the well-established least squares or maximum likelihood techniques fail (Weise and Wöger, 1993; Weise and Wöger, 1994). It offers a very clear insight into robust estimation. Robustness can be directly set up by a suitable design of probability distribution of the measured data. The approach can also be utilized in collocation. Hence, collocation can be explained from the Bayesian viewpoint (Soukup, 2001). By using Bayesian approach, determination of ambiguities in phase observations of GPS and deriving confidence regions are conducted. For these applications see (Grodecki, 2001; Gundlich, 2002; Gundlich and Koch, 2002; Lacy et all, 2002, Zhu et all, 2001). Bayesian approach is used as a statistical approach in digital image analysis. For this application see (Ding, 2002).

6. Conclusions

We performed this study to investigate of applicability of Bayesian approach in uncertainty assessment in geodetic deformation analysis problems. For this aim we have constituted background information about Bayesian Statistics and uncertainty assessment, and investigated different applications in Geodesy using Bayesian statistics. Bayesian statistical methods have become powerful and efficient tools in diverse areas of statistics, particularly medicine, social studies, archeology, etc. The Bayesian approach to statistics provides a unified framework for optimally combining information from multiple sources and for incorporating previous experimental results and/or expert opinion into current experimentation and modeling. This results in simpler, highly efficient experimental designs and statistical analyses.

The difference between Bayesian and frequentist approach is the way that parameters in the model are treated. The frequentist treats the parameters are fixed quantities while the Bayesian allows the parameters to be random variables. In classical methods, stochastic parts are usually modeled as a simple random variable, which adds only noise to the deterministic part. Recent interest in statistics has focused on their use for applications involving non-normal data. The most common analysis of statistical uncertainty is based on the “likelihood” function. The likelihood function is a function relating observations to parameters. Mathematically this approach is useful in estimation and testing problems. A common approach to estimation is to choose the values of parameters that make the data as likely as possible (MLE). The view just described is the classical or frequentist view of statistics. A different view is given Bayesian method (Smith, 2002).

In our future studies, according to introduced theoretical knowledge about Bayesian approach, we plan to apply Bayesian approach in uncertainty assessment of geodetic data. For this aim, firstly, we plan to constitute Gaussian distribution for data as an expression of prior knowledge. Secondly, we intend to form other distributions for data as an expression of prior knowledge. We will also solve this problem by classical methods, LSE and MLE. After these studies, we intend to compare results. By these studies, we can take an opportunity to compare classical and Bayesian approach in the processing steps and results.

References

Berger, J.O. (1985). *Statistical Decision Theory and Bayesian Analysis*, ISBN 0-387-96098-8 Springer-Verlag, New York, Berlin, Heidelberg.

Bernardo J. M., A.F.M. Smith. (1994). *Bayesian Theory*, John Wiley & Sons, ISBN 0 471 92416 4, England.

Bod, R. (2001). *Introduction to Probability Theory in Linguistics*, LSA-2001 Workshop Probability Theory in Linguistics, Linguistic Society of America 2001, Washington D.C.

Box, G. E. P., G. C. Tiao (1992). *Bayesian Inference in Statistical Analysis*, A Wiley-Interscience Publication, ISBN 0 471 57428 7, USA.

Carlson, G.A. (2002). *Experimental Errors and Uncertainty*, http://www.stchas.edu/faculty/gcarlson/physics/docs/Error_and_Uncertainty.pdf

Caspary, W., H. Borutta (1986). Geometrische Deformationsanalyse mit Robusten Sch”atzverfahren, Allgemeine Vermessungsnachrichten 8-9/1986, 315-326

Caspary, W.F., W. Haen., H. Borutta (1990). Deformation Analysis by Statistical Methods, *Technometrics*,Vol.32, No.1

Chen, S., E. Nikolaidis, H. Cudney (1999**).** Comparision of Probability and Fuzzy Models for Desinging under Uncertainty, American Inst. of Aeronautics and Astronautics, AIAA-99-1579.

Crovelli, A. R. (2000) Probability Models for Estimation of Number and Costs of Landslides, *U.S. Geological Survey (USGS), Open-File Report* 00-249, MS 939, Box 25046, Denver Federal Center, Denver, Colorado 80225.

D'Agostini, G. (1995). Probability and Measurement Uncertainty in Physics- a Bayesian Primer, Notes based on lectures to the DESY summer students (September 1995), DESY 95-242.

D'Agostini G. (2003). Bayesian Inference in Processing Experimental Data: Principles and Basic Applications, arXiv.org e-Print archive, Physics, 0304102v1.

Dean, T. (2002). Introduction to Probability and Reasoning Under Uncertainty, Brown University Computer Science, Courses (Building Intelligent Robots), http://www.cs.brown.edu/people/tld/courses/cs148/02/supplements/Lecture.Probability.pdf

Dilaver, A. (1996). Detection of Outliers in Observations and Reliablity Criteria of Geodetic Networks, Karadeniz Technical University, Department of Geodesy and Photogrammetry Engineering Research Reports, 1996, Trabzon, Turkey (in Turkish).

Ding, M. (2002). Seminer Series on Advanced Medical Image Processing, Robarts Research Institute, London, Ontario, Canada,

Gelman, A., J. B. Carlin, H.S. Stern, D.B. Rubin (1995). *Bayesian Data Analysis*, Chapman& Hall, ISBN 0 412 03991 5, 2-6 Boundary Row, London SE1 8HN,UK.

Grodecki, J. (2001). Generalized Maximum –Likelihood Estimation of Variance-Covariance Components with Non-Informative Prior, *Journal of Geodesy* , 75, 157-163.

Gundlich, B., K.R. Koch (2002). Confidence Regions for GPS Baselines by Bayesian Statistics, *Journal of Geodesy*, 76, 55-62.

Hekimoglu, S., H. Demirel and C. Aydın (2002). Reliability of the Conventional Deformation Analysis Methods for Vertical Networks, *FIG XXII Int. Cong*., Washington, D.C. USA.

Isukapalli, S. S. (1999). Uncertainty Analysis of Transport-Transformation Models, A dissertation submitted to the Graduate School--New Brunswick Rutgers, The State University of New Jersey, New Brunswick, New Jersey.

Kutterer, H. (2001a). Non-Probabilistic Assessment in Geodetic Data Analysis*, IAG Scientific Assembly* 2001, Budabest.

Kutterer, H. (2001b). Uncertainty Assessment in Geodetic Data Analysis, IAG, *First International Symposium on Robust Statistics and Fuzzy Techniques* in Geodesy and GIS, Zurich, Switzerland.

Koch, K. R. (1990). *Bayesian Inference with Geodetic Applications*, Springer, Berlin, Heidelberg, New York.

Koch, K. R. (2000). Some Basics of Bayesian Statistics, *Geodesy and Cartography*, Vol XXVI, No.4.

Koch, K. R. (1999). Bayesian Networks for Decision in Systems with Uncertainties*, Third Turkish -German Joint Geodetic Days*, Ístanbul, Türkiye.

Lacy, M. C. De, F. Sanso, G. Rodriguez-Caderot, A.J. Gill (2002), The Bayesian Approach Applied to GPS Ambiguity Resolution. A Mixture Model for The Discrete- Real Ambiguites Alternative, *Journal of Geodesy*, 76, 82-94.

Pres, S. J. (1989). *Bayesian Statistics: Principles, Models, and Applications*, John Wiley & Sons, ISBN 0 471 63729 7, USA.

Robiah A., S. Halim, and M. Mohd Nor (2001). Identifying Multiple Outliers in Linear Regression: Robust Fit and Clustering Approach, *10th FIG Int. Symposium on Deformation Measurements*, Orange, California, USA.

Shyllon, E. A. (2001). Fuzzy System as Applied to Geodetic Data Analysis, *IAG 2001* Scientific Assembly, Budabest, 02-08.09.2001.

Smith, E. (2002). Uncertainty Analysis, Volume 4, pp 2283–2297 *in Encyclopedia of Environmetrics*, ISBN 0471 899976, Edited by, Abdel H. El-Shaarawi and Walter W. Piegorsch, John Wiley & Sons, Ltd, Chichester, 2002.

Soukup L. (2001). Least Squares Without Minimization, *IAG Scientific Assembly 2001*, Budabest.

Weise, K., W. Wöger (1994). Comparision of Two Measurement Results using The Bayesian Theory of Measurement Uncertainty, *Meas. Sci. Technol.* , 5, UK.

Weise, K., W. Wöger (1993). A Bayesian Theory of Measurement Uncertainty*, Measurement Science and Technology,* 4, UK.

Wöger, W. (2001). Assigning Probability Distribution Functions (PDF) to Input Quantities, Physikalisch-Technische Bundesanstalt (PTB) Bundesallee 100, D-38116 Braunschweig, Germany.

Zeitler, J. (1999). *Bayesian and Classical Statistics*, http://mhhe.com/business/opsci/aczel/textfigures/chap15/index.htm

Zhu, J., X. Ding, Y. Chen (2001). Maximum–Likelihood Ambiguity Resolution Based on Bayesian Principle, *Journal of Geodesy* , 75, 175-187.